Homogeneous dynamics and its applications to number theory

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Based on joint work with **Timothée Bénard** and **Weikun He**

Seminar of Analysis and Probability at Wuhan University

Outline of the talk

- Introduction to Homogeneous dynamics.
  (1) An motivating example on $\mathbb{T}^2$.
  (2) Ratner's uniform equidistribution theorem: qualitative and effective aspects.
- Some applications of effective aspects of homogeneous dynamics to number theory
  (1) Oppenheim's conjecture on quadratic forms.
  (2) Khintchine's theorem in Diophantine approximation.
- Some further questions.

An example in $\mathbb{T}^2$

A **motivating** example:

Denote by $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ a 2-dimensional torus.
For a vector $\boldsymbol{v} \in \mathbb{R}^2$, the orbit $\{t\boldsymbol{v} \bmod \mathbb{Z}^2 \in \mathbb{T}^2 : t \in \mathbb{R}\}$ is

$$\begin{cases} \textbf{periodic,} & \text{if the slope of } \boldsymbol{v} \text{ is } \textbf{rational}, \\ \textbf{equidistributed,} & \text{if the slope of } \boldsymbol{v} \text{ is } \textbf{irrational}, \end{cases}$$

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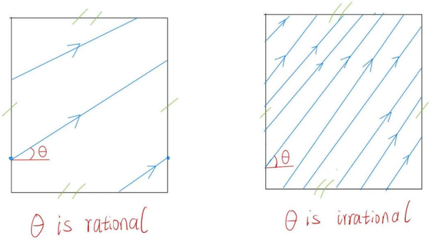

Figure: Equidistribution on $\mathbb{T}^2$

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- $G = \mathbb{R}^2 =$ a Lie group.
- $\Gamma = \mathbb{Z}^2 =$a lattice of $\mathbb{R}^2$.
- $\mathbb{T}^2 = G/\Gamma =$a homogeneous space of $\mathbb{R}^2$.

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- $U = \{t\boldsymbol{v} \in \mathbb{R}^2 : t \in \mathbb{R}\}$= a one-parameter **unipotent** subgroup in the Lie group $\mathbb{R}^2$.

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Question: Let G be a Lie group, $\Gamma < G$ a lattice, $U < G$ a unipotent subgroup and $x \in G/\Gamma$. Do we have a nice description of $Ux \subset G/\Gamma$?

Homogeneous dynamics: qualitative aspect

Dani (1982): Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Consider

$$U = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}.$$

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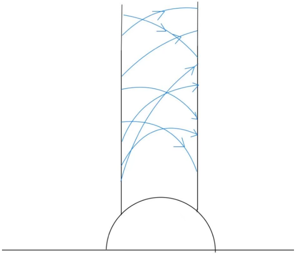

Figure: Equidistribution of Ux in the upper half plane model of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$

Homogeneous dynamics: qualitative aspect

Raghunathan's conjecture (Mid-1970s): Let G be a Lie group, $\Gamma < G$ be a lattice and $U < G$ be a one-parameter unipotent subgroup of G. Then for any $x \in G/\Gamma$, the orbit Ux is **equidistributed** in the smallest sub-homogeneous space in G/Γ containing Ux.

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In applications to number theory: Shah's result on equidistribution of **expanding translates** of a unipotent orbit by a diagonal flow.

Homogeneous dynamics: qualitative aspect

Theorem (Shah, 1996, special case 1)

Let $G = \mathrm{SL}_{d+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$. *For* $t > 0, \boldsymbol{s} \in \mathbb{R}^d$, *let*

$$a(t) = \begin{pmatrix} t^{\frac{1}{d+1}} & & & \\ & \ddots & & \\ & & t^{\frac{1}{d+1}} & \\ & & & t^{-\frac{d}{d+1}} \end{pmatrix}, \quad u(\boldsymbol{s}) = \begin{pmatrix} 1 & & & s_1 \\ & \ddots & & \vdots \\ & & 1 & s_d \\ & & & 1 \end{pmatrix}.$$

Then for any $x \in G/\Gamma$, *any* $f \in C_c(G/\Gamma)$,

$$\lim_{t\to\infty} \int_{[0,1]^d} f(a(t)u(\boldsymbol{s})x)\mathrm{d}\boldsymbol{s} = \int f \mathrm{d}m_{G/\Gamma}.$$

Remark: G/Γ= the space of all unimodular lattices in $\mathbb{R}^{d+1}$.

Homogeneous dynamics: qualitative aspect

Theorem (Shah, 1996, special case 2)

Let $G = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$, $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ *and* $H = \mathrm{SL}_2(\mathbb{R}) \times \{\mathbf{0}\}$. *For any* $t > 0$ *and* $s \in \mathbb{R}$, *let*

$$a(t) = \left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2} \end{pmatrix}, \mathbf{0} \right), u(s) = \left(\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}, \mathbf{0} \right).$$

Then for any $y \in G/\Gamma$, *either* Hy *is periodic or for any* $f \in C_c(G/\Gamma)$,

$$\lim_{t\to\infty} \int_0^1 f(a(t)u(s)y)\mathrm{d}s = \int f \mathrm{d}m_{G/\Gamma}.$$

Example: For $y_{\boldsymbol{\xi}} = (\mathrm{Id}, \boldsymbol{\xi})\Gamma/\Gamma$ where $\boldsymbol{\xi} \in \mathbb{R}^2$, Hy is periodic if and only if $\boldsymbol{\xi} \in \mathbb{Q}^2$.

Remark: G/Γ= the space of affine unimodular lattices in $\mathbb{R}^2$.

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Ratner's and Shah's uniform distribution theorem is **qualitative**.

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Examine dynamics in $\mathbb{T}^2$:

Theorem (Weyl's effective equidistribution theorem)

Given $\boldsymbol{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *with slope* θ. *Assume* θ *is Diophantine, then there exists* $c = c(\theta) > 0$ *such that for any* $f \in C^\infty(\mathbb{T}^2)$,

$$\frac{1}{T}\int_0^T f(t\boldsymbol{v})\mathrm{d}t = \int f \mathrm{d}m_{\mathbb{T}^2} + O(T^{-c})\mathcal{S}(f).$$

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In order to obtain more powerful results in number theory, an **effective** Ratner/Shah-type theorem is desired.

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Theorem (Kleinbock-Margulis, 1996)

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Proof: using exponential mixing of $a(t)$ action+Margulis' thickening trick.

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Given $y_\xi = (\mathrm{Id}, \xi)\Gamma/\Gamma$ such that ξ is Diophantine. Then there exists $c = c(\xi) > 0$ such that for any $f \in C_c^\infty(G/\Gamma)$ and $t > 0$,

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Proof: using very delicate Fourier analysis on the torus fiber bundle $\mathbb{T}^2$.

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Bénard-He (2024): Effective Ratner-type theorem for semisimple random walks on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

Bénard-He-Zhang (2024,2025): Effective Ratner-type theorem for some upper triangular random walks on $\mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$.

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Remark: Only **a few** special cases of Oppenheim's conjecture were proved using analytic number theory method before Margulis.

Oppenheim's conjecture

Using effective Ratner's theorem in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$:

Theorem (Lindenstrauss-Mohammadi-Wang-Yang, 2025)

Given a non-degenerated, indefinite and irrational quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$. Assume that Q is "badly approximable" by all rational quadratic forms. Then there exists $\kappa = \kappa(Q) > 0, c_Q > 0$ such that for any $(a, b) \subset \mathbb{R}$,

$$\#\{\boldsymbol{v} \in \mathbb{Z}^3 : \|\boldsymbol{v}\| \leq T, Q(\boldsymbol{v}) \in (a, b)\} = \underbrace{c_Q(b-a)T + \mathcal{R}(T)}_{\text{main term}} + \underbrace{O_{a,b}(T^{1-\kappa})}_{\text{error}}.$$

Khintchine's theorem

Let $d \geq 1$ be an integer. Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a **non-increasing** function. A vector $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$ is called ψ-**approximable** if there exist **infinitely** many $(\boldsymbol{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ such that

$$\|q\boldsymbol{x} - \boldsymbol{p}\|_\infty < \psi(q).$$

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A main goal: Study how the **size** of $W(\psi)$ depend on ψ.

Khintchine's theorem

Theorem (Khintchine, 1920s)

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *be a* **non-increasing** *function. Denote by* $\mathrm{Leb}_{[0,1]^d}$ *the Lebesgue measure on* $[0,1]^d$*. Then*

$$\mathrm{Leb}_{[0,1]^d}(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q)^d < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q)^d = \infty. \end{cases}$$

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Theorem (Schmidt, 1960)

For Leb-*a.e.* $\boldsymbol{s} \in \mathbb{R}^d$*, as* $n \to +\infty$*:*

$$\left|\{(\boldsymbol{p}, q) \in \mathbb{Z}^d \times \mathbb{N} \,:\, \|q\boldsymbol{s} - \boldsymbol{p}\|_\infty < \psi(q), 1 \leq q \leq n\}\right| \sim_{\boldsymbol{s},\psi} 2^d \sum_{q=0}^{n} \psi(q)^d.$$

Khintchine's theorem

Kleinbock-Margulis (1999): an alternative proof of the classical Khintchine's theorem using **effective** dynamics in $\mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$: for all $f \in C_c^\infty(G/\Gamma)$, $t > 0$ and $x \in X$

$$\int_{[0,1]^d} f(a(t)u(\boldsymbol{s})x)\mathrm{d}\boldsymbol{s} = \int f \mathrm{d}m_{G/\Gamma} + O(t^{-c}\operatorname{inj}(x)^{-1})\mathcal{S}(f),$$

where

$$a(t) = \begin{pmatrix} t^{\frac{1}{d+1}} & & & \\ & \ddots & & \\ & & t^{\frac{1}{d+1}} & \\ & & & t^{-\frac{d}{d+1}} \end{pmatrix}, \quad u(\boldsymbol{s}) = \begin{pmatrix} 1 & & & s_1 \\ & \ddots & & \vdots \\ & & 1 & s_d \\ & & & 1 \end{pmatrix}.$$

Khintchine's theorem for self-similar measures

Theorem (BHZ, 2024,2025)

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *be a* **non-increasing** *function. Let* σ *be a self-similar measure on* $\mathbb{R}^d$*. Then*

$$\sigma(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q)^d < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q)^d = \infty. \end{cases}$$

Moreover, for σ*-a.e.* $\boldsymbol{s} \in \mathbb{R}^d$*, as* $n \to +\infty$*, we have*

$$\left|\{(\boldsymbol{p}, q) \in \mathbb{Z}^d \times \mathbb{N} \, : \, \|q\boldsymbol{s} - \boldsymbol{p}\|_\infty < \psi(q), 1 \leq q \leq n\}\right| \sim_{\boldsymbol{s},\psi} 2^d \sum_{q=0}^{n} \psi(q)^d.$$

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Previous results on Khintchine's theorem for self-similar measures: Weiss (2000), Kleinbock-Lindenstrauss-Weiss (2004), Einsiedler-Fishman-Shapira (2011), Simmons-Weiss (2019), Yu (2021) Khalil-Luethi (2023), Datta-Jana (2024).

Khintchine's theorem for self-similar measures

Approach to Khintchine's theorem for self-similar measures: an effective Kleinbock-Margulis equidistribution theorem for self-similar measures.

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Let $G = \mathrm{SL}_{d+1}(\mathbb{R}), \Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$ and σ a self-similar measure on $\mathbb{R}^d$. Then there exists $c = c(\sigma) > 0$ such that for all $f \in C_c^\infty(G/\Gamma)$, $t > 0$ and $x \in X$

$$\int f(a(t)u(\boldsymbol{s})x)\mathrm{d}\sigma(\boldsymbol{s}) = \int f \mathrm{d}m_{G/\Gamma} + O(t^{-c}\,\mathrm{inj}(x)^{-1})\mathcal{S}(f),$$

where $a(t) = \begin{pmatrix} t^{\frac{1}{d+1}} & & & \\ & \ddots & & \\ & & t^{\frac{1}{d+1}} & \\ & & & t^{-\frac{d}{d+1}} \end{pmatrix}, \quad u(\boldsymbol{s}) = \begin{pmatrix} 1 & & & s_1 \\ & \ddots & & \vdots \\ & & 1 & s_d \\ & & & 1 \end{pmatrix}.$

Some further questions on homogeneous dynamics and Diophantine approximation

- Beresnevich-Velani: Prove a Jarník-Besicovitch result for self-similar fractals: Given a non-increasing ψ and a self-similar $\mathcal{K} \subset \mathbb{R}^d$, compute

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- Khintchine's theorem for self-affine measures and self-conformal measures: may requires one to study the random walks induced by self-affine/conformal IFS on homogeneous spaces.

Thanks for your attention!

Any questions ?