Multifractal Analysis of Spectral Measures for Sturmian Hamiltonians and the Almost Mathieu Operator

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WuHan University October 27, 2025

Discrete Schrödinger operator

Given $V:\mathbb{Z}\to\mathbb{R}$ bounded. Define the discrete Schrödinger operators $H_V:\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ as

$$\begin{aligned} H_V\psi &:= \Delta\psi + V\psi \\ (H_V\psi)_n &:= (\psi_{n+1}+\psi_{n-1}) + V_n\psi_n. \end{aligned}$$

Fact: H_V is bounded, self-adjoint, the spectrum $\sigma(H_V)\subset\mathbb{R}$ is compact.
Physically: It describe the motion of an electron in a material.
The spectral property is related to the conductivity of the material.

Spectral measure

For any $\psi \in \ell^2(\mathbb{Z})$, the spectral measure μ_ψ is defined by (via Riesz presentation theorem)

$$\int_{\sigma(H_V)} f(E) d\mu_\psi(E) := \langle \psi, f(H_V)\psi \rangle, \quad f \in C(\sigma(H_V)).$$

Define the spectral measure of H_V as

$$\mu_V := \frac{\mu_{\delta_0} + \mu_{\delta_1}}{2}.$$

Fact: For any $\psi \in \ell^2(\mathbb{Z})$, one has $\mu_\psi \ll \mu_V$.
Physically: If μ_V is a.c. (p.p., "s.c.") then the material is a conductor (insulator, "semi-conductor")

Periodic potential case— Floquet-Bloch theory

Theorem (Floquet-Bloch)

Assume V is n-periodic, then the spectrum of H_V is given by

$$\sigma(H_V) = \{E \in \mathbb{R} : |t_V(E)| \le 2\} = B_1 \cup B_2 \cup \cdots \cup B_n,$$

where t_V is a polynomial of degree n, called the trace polynomial of H_V. The spectral measure $\mu_V \ll \mathcal{L}|_{\sigma(H_V)}$.

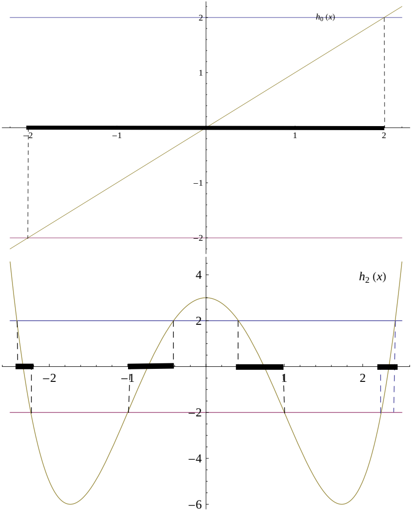
- For $V \equiv 0$. We have

$$t_0(E) = E; \quad \sigma(H_0) = [-2, 2]; \quad \mu_0 = \frac{\chi_{[-2,2]}(E)dE}{\pi\sqrt{4 - E^2}}$$

- V is 4-periodic and

$$V|_{[1,4]} = (1, -1, -1, 1); \quad t(E) = E^4 - 6E^2 + 3.$$

Pictures of the Spectra



Quasi-periodic potentials

Two classes of quasi-periodic potentials are heavily studied, they all have the following form:

$$V_{f,\alpha,\lambda,\theta}(n) = \lambda f(\theta + n\alpha) \qquad (1)$$

where $f\colon \mathbb{S}^1 \to \mathbb{R}$ is bounded, $\alpha \in [0,1] \setminus \mathbb{Q}$, $\lambda > 0$ and $\theta \in \mathbb{S}^1$.

- Almost Mathieu potential:

$$f(x) = 2cos2\pi x.$$

  The related operator is called AMO.

- Sturmian potential:

$$f(x) = \chi_{[1-\alpha,1)}(x).$$

  The related operator is called Sturmian Hamiltonian.

Spectrum and density of states

For operator with potential (1), by the general theory of ergodic Schrödinger operators, the spectrum is independent of θ. So we write

$$\Sigma_{\alpha,\lambda}^{f} := \sigma(H_{V_{f,\alpha,\lambda,\theta}}).$$

Another important measure, called density of states (DOS) of the operator, is defined as the average of the spectral measures:

$$\mathcal{N}_{\alpha,\lambda}^{f} := \int_{\mathbb{S}^1} \mu_{V_{f,\alpha,\lambda,\theta}} d\theta.$$

Now we focus on Sturmian Hamiltonian and simply the notions to

$$H_{\alpha,\lambda,\theta}, \quad \Sigma_{\alpha,\lambda}, \quad \mathcal{N}_{\alpha,\lambda}.$$

Cantor spectrum–fractal is coming

To study quasi-periodic operators, we do the periodic approximation: Choose potentials $V^{(n)}$ which is k_n-periodic such that $V^{(n)} \to V$ in suitable sense. Then $H_n := H_{V^{(n)}} \xrightarrow{s} H_V$. As a consequence,

$$d_H(\sigma(H_n), \sigma(H_V)) \to 0.$$

By Floquet-Bloch theory, $\sigma(H_n)$ is made of k_n non-overlapping bands. When $n \to \infty$, the spectrum has the tendency to be a Cantor set.

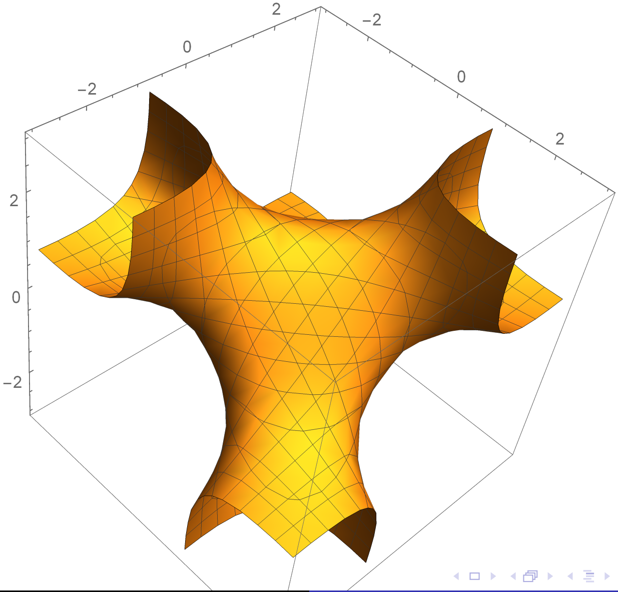
Deterministic results

• Fibonacci Hamiltonian:The operator $H_{\alpha_1,\lambda,\theta}$ with golden ratio $\alpha_1 := (\sqrt{5}+1)/2$. This model was introduced by Kohmoto et. al. and Ostlund et. al.(1983) as a model for quasicrystal.
Define the Fibonacci trace map $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\mathbf{T}(x, y, z) := (2xy - z, x, y).$$

Then $G(x, y, z) := x^2 + y^2 + z^2 - 2xyz - 1$ is invariant under $\mathbf{T}$. So for $\lambda > 0$, $\mathbf{T}$ preserves the cubic surface

$$S_\lambda := \{(x, y, z) \in \mathbb{R}^3 : G(x, y, z) = \lambda^2/4\}.$$


2
0
−2
−2
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2
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−2

Fibonacci Hamiltonian

Write $\mathbf{T}_\lambda := \mathbf{T}|_{S_\lambda}$ and let Λ_λ be the attractor of $\mathbf{T}_\lambda$. Then Λ_λ is a locally maximal compact transitive hyperbolic set of $\mathbf{T}_\lambda$.

Theorem (Casdagli (CMP 1986), Sütö (CMP 1987), $\cdots$, Damanik-Gorodetski-Yessen (Invent 2016))

For Fibonacci Hamiltonian, the following hold:
1) The spectrum $\Sigma_{\alpha_1,\lambda}$ satisfies

$$\dim_H \Sigma_{\alpha_1,\lambda} = \dim_B \Sigma_{\alpha_1,\lambda} =: D(\alpha_1, \lambda).$$

2) $D(\alpha_1, \lambda)$ satisfies Bowen's formula: $D(\alpha_1, \lambda)$ solves the equation $P(t\phi_\lambda) = 0$, where ϕ_λ is the geometric potential on Λ_λ

$$\phi_\lambda(x) := -\log \|D\mathbf{T}_\lambda(x)|_{E^u}\|.$$

Theorem (continued)

3) The DOS $\mathcal{N}_{\alpha_1,\lambda}$ is exact-dimensional and consequently

$$\dim_H \mathcal{N}_{\alpha_1,\lambda} = \dim_P \mathcal{N}_{\alpha_1,\lambda} =: d(\alpha_1, \lambda).$$

4) $d(\alpha_1, \lambda)$ satisfies Ledrappier-Young's formula:

$$d(\alpha_1, \lambda) = \dim_H \mu_{\lambda,\max} = \frac{\log \alpha_1}{\mathrm{Lyap}^u \mu_{\lambda,\max}},$$

where $\mu_{\lambda,\max}$ is the measure of maximal entropy of $\mathbf{T}_\lambda$, and $\log \alpha_1, \mathrm{Lyap}^u \mu_{\lambda,\max}$ are the entropy and the unstable Lyapunov exponent of $\mu_{\lambda,\max}$, respectively.

5) $d(\alpha_1, \lambda) < D(\alpha_1, \lambda)$.(Barry Simon's Conjecture)

The coding of the spectra

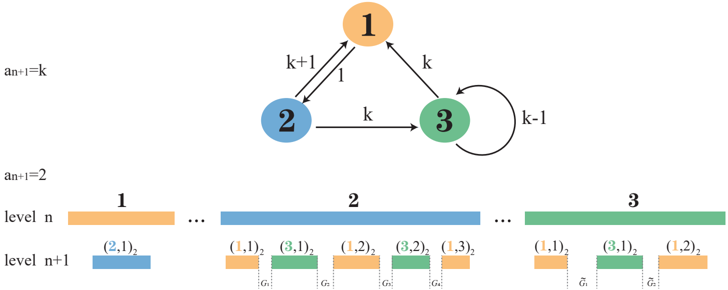
For the spectrum of Sturmain Hamiltonian, we have the following very explicit coding of the established by Raymond:

Theorem (Raymond 1997 (Preprint))

For any $\lambda > 4$ and $\alpha \in [0,1] \setminus \mathbb{Q}$, there exists a symbolic space Ω_α and a coding map $\pi_\alpha : \Omega_\alpha \to \Sigma_{\alpha,\lambda}$.

For Fibonacci Hamiltonian, the symbolic space Ω_{α_1} is essentially the subshift of finite type with alphabet $\mathcal{A} := \{e_1, e_2, e_3, e_4\}$ and coincidence matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$


an+1=k
1
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an+1=2
1
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level n
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(2,1)2
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(3,1)2
(1,2)2

Sturmian Hamiltonian-specturm

Assume $\alpha \in [0,1] \setminus \mathbb{Q}$ has expansion $\alpha = [0; a_1, a_2, \cdots]$. Define

$$K_*(\alpha) = \liminf_{n\to\infty} (\prod_{j=1}^{n} a_j)^{1/n}; \quad K^*(\alpha) = \limsup_{n\to\infty} (\prod_{j=1}^{n} a_j)^{1/n}.$$

Theorem (Liu-Wen (Potential 2004),$\cdots$,Liu-Qu-Wen (Adv 2014))

1) Assume $\lambda \geq 24$. The following dichotomies hold:

$$\begin{cases} \dim_H \Sigma_{\alpha,\lambda} \in (0,1) & \text{if} \quad K_*(\alpha) < \infty \\ \dim_H \Sigma_{\alpha,\lambda} = 1 & \text{if} \quad K_*(\alpha) = \infty \end{cases}$$

$$\begin{cases} \overline{\dim}_B \Sigma_{\alpha,\lambda} \in (0,1) & \text{if} \quad K^*(\alpha) < \infty \\ \overline{\dim}_B \Sigma_{\alpha,\lambda} = 1 & \text{if} \quad K^*(\alpha) = \infty \end{cases}.$$

Theorem (continued)

2) $\underline{D}(\alpha,\cdot)$ *and* $\overline{D}(\alpha,\cdot)$ *are Lipschitz continuous on any bounded interval of* $[24,\infty)$ *such that*

$$\underline{D}(\alpha,\lambda)=\dim_H\Sigma_{\alpha,\lambda} \quad \textit{and} \quad \overline{D}(\alpha,\lambda)=\overline{\dim}_B\Sigma_{\alpha,\lambda}.$$

Here, $\underline{D}(\alpha,\cdot)$ and $\overline{D}(\alpha,\cdot)$ are the pre-dimensions of $\Sigma_{\alpha,\lambda}$:

$$\underline{D}(\alpha,\lambda)=\liminf_{n\to\infty} s_n(\alpha,\lambda); \quad \overline{D}(\alpha,\lambda)=\limsup_{n\to\infty} s_n(\alpha,\lambda),$$

where $s_n(\alpha,\lambda)$ is the unique number such that

$$\sum_{w\in\Omega_{\alpha,n}}|B_w^\alpha(\lambda)|^{s_n(\alpha,\lambda)}=1.$$

Sturmian Hamiltonian-DOS

Theorem (Qu, IMRN 2018)

For any $\lambda > 24$, $\alpha = [a_1, a_2, \cdots]$ *with* $a_k \le M, k \in \mathbb{N}$. *The DOS* $\mathcal{N}_{\alpha,\lambda}$ *is both exact upper- and lower-dimensional. there exists a certain* α *such that the related* $\mathcal{N}_{\alpha,\lambda}$ *is not exact-dimensional.*

Theorem (Jitomirskaya-Zhang, JEMS 2022)

For any $\lambda > 0$, *there exists Liouvillian frequency* α *such that the related DOS satisfies* $\dim_H \mathcal{N}_{\alpha,\lambda} < 1$ *but* $\dim_P \mathcal{N}_{\alpha,\lambda} = 1$. *Consequently,* $\mathcal{N}_{\alpha,\lambda}$ *is not exact-dimensional.*

Bellissard's conjecture and Damanik-Gorodetski's result

Until now, all the results are stated for deterministic frequencies. How about the dimensional properties of $\Sigma_{\alpha,\lambda}$ and $\mathcal{N}_{\alpha,\lambda}$ for Leb. typical frequency?
Bellissard had the following conjecture in 1980s:

Conjecture(Bellissard 1980s): For every $\lambda > 0$, the Hausdorff dimension of $\Sigma_{\alpha,\lambda}$ is Leb. a.e. constant in α.

Theorem (Damanik-Gorodetski, CMP 2015)

For every $\lambda \geq 24$, there exists two numbers $0 < \underline{D}(\lambda) \leq \overline{D}(\lambda)$ such that for Lebesgue almost every $\alpha \in [0,1] \setminus \mathbb{Q}$,

$$\dim_H \Sigma_{\alpha,\lambda} = \underline{D}(\lambda) \quad \textit{and} \quad \overline{\dim}_B \Sigma_{\alpha,\lambda} = \overline{D}(\lambda).$$

Idea of the proof(Based on Liu-Qu-Wen 2014): Show that $\underline{D}(\cdot,\lambda)$ is measurable and invariant under Gauss measure G. Then use the ergodicity of G. The same for $\overline{D}(\cdot,\lambda)$.

Natural questions: For fixed $\lambda \geq 24$, whether $\underline{D}(\lambda) = \overline{D}(\lambda)$ holds? Does the full measure set of frequencies depend on λ? How regular are the functions $\underline{D}(\lambda)$ and $\overline{D}(\lambda)$? What can one say about the DOS? etc.

a.s. dimensional properties of the spectrum and the DOS

Theorem (C-Qu, Adv 2025)

There exist a subset $\mathbb{I} \subset [0,1] \setminus \mathbb{Q}$ of full Lebesgue measure and two functions $d, D : [24, \infty) \to (0,1)$ such that
1) For any $(\alpha, \lambda) \in \mathbb{I} \times [24, \infty)$, the spectrum $\Sigma_{\alpha,\lambda}$ satisfies

$$\dim_H \Sigma_{\alpha,\lambda} = \dim_B \Sigma_{\alpha,\lambda} = D(\lambda).$$

Moerover, $D(\lambda)$ satisfies a Bowen type formula: $D(\lambda)$ is the unique zero of a relativized pressure function $P_G(\Psi^*_{t,\lambda})$.

$$P_G(\Psi^*_{\lambda,t}) := \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{N}^{\mathbb{N}}} \psi^*_{\lambda,t,n}(\alpha) G(d\alpha).$$

Theorem (continued)

2) For any $(\alpha, \lambda) \in \mathbb{I} \times [24, \infty)$, $\mathcal{N}_{\alpha,\lambda}$ is exact-dimensional and

$$\dim_H \mathcal{N}_{\alpha,\lambda} = \dim_P \mathcal{N}_{\alpha,\lambda} = d(\lambda).$$

Moreover, $d(\lambda)$ satisfies a Ledrappier-Young type formula:

$$d(\lambda) = \frac{\gamma}{-(\Psi_\lambda)_*(\mathscr{N})},$$

where γ is the Lévy's constant, $\mathscr{N}$ is a Gibbs measure on the global symbolic space Ω.

Here, $\Psi_\lambda := \{\psi_{\lambda,n} : \ n \geq 1\}$, $\psi_{\lambda,n}(x) = \log |B^\alpha_{x|_n}(\lambda)|$ if $\pi(x) = \alpha$ and $\psi^*_{\lambda,t,n}(\alpha) := \log \sum_{w \in \Omega_{\alpha,n}} \exp(t\psi_{\lambda,n}(x_w)) = \log \sum_{w \in \Omega_{\alpha,n}} |B^\alpha_w(\lambda)|$.

(Here, $|B^\alpha_{x|_n}(\lambda)|$ is a covering band of order n)

Natural questions: For fixed $\lambda \geq 24$, whether $d(\lambda) < D(\lambda)$ holds? Can we use the DOS $\mathcal{N}_{\alpha,\lambda}$ to provide a hierarchical characterization of the specturm $\Sigma_{\alpha,\lambda}$? That is, to study the Hausdorff dimension of the level set $\Sigma_{\alpha,\lambda}(\kappa)$ of $\mathcal{N}_{\alpha,\lambda}$,

$$\Sigma_{\alpha,\lambda}(\kappa) = \left\{ x \in \Sigma_{\alpha,\lambda} : \lim_{r \to 0^+} \frac{\log \mathcal{N}_{\alpha,\lambda}(B(x,r))}{\log r} = \kappa \right\}.$$

Furthermore, does the multifractal formalism holds in this case?

a.s. multifractal formalism for the DOS

Theorem (C-Qu 2025)

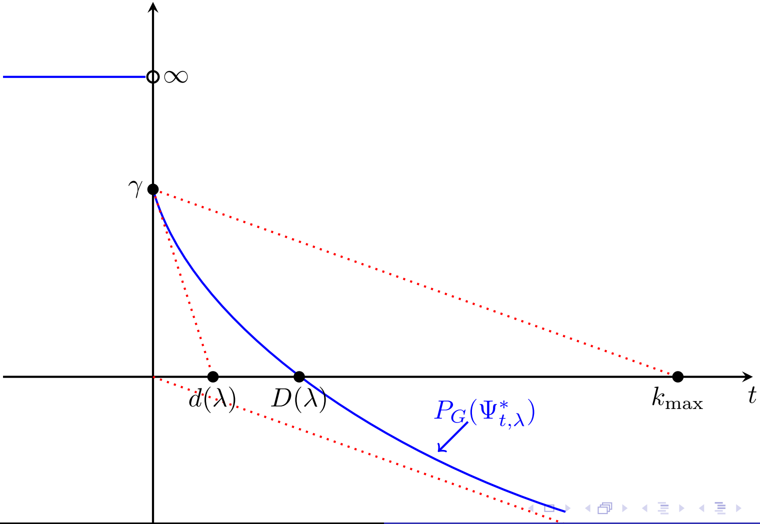
Fix each $\lambda \geq 24$, the following hold:
1) The function $t \mapsto P_G(\Psi^*_{\lambda,t})$ is C^1, strictly convex on $[0,\infty)$.
2) $d(\lambda) < D(\lambda)$.
3) For any $(\alpha,\kappa) \in \mathbb{I} \times [0, k_{\max}]$,

$$\dim_H \Sigma_{\alpha,\lambda}(\kappa) = \inf_t \left\{ P_G(\Psi^*_{t,\lambda}) + \kappa t \right\},$$

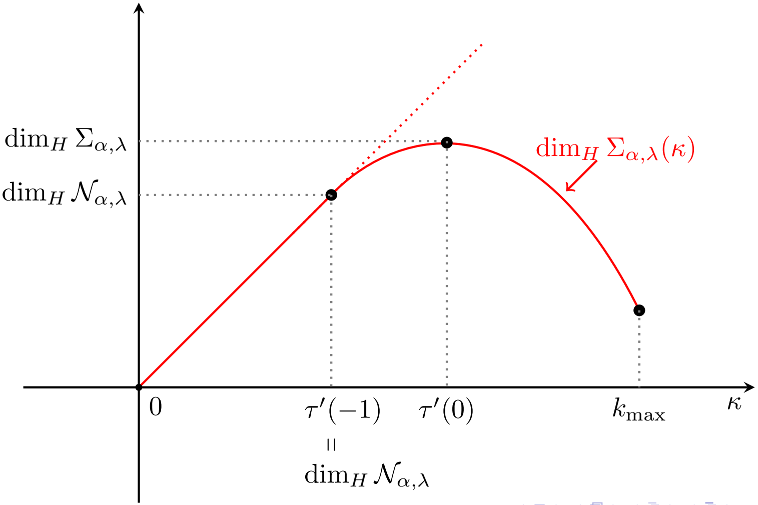
where $\gamma/k_{\max} = -\lim_{n\to\infty} \frac{1}{n} \sup\{|B^{\alpha}_{w}(\lambda)| : w \in \Omega_{\alpha,n}\}$.

The particular inequality $dim_H \mathcal{N}_{\alpha,\lambda} < dim_H \Sigma_{\alpha,\lambda}$ establishes a conjecture of Barry Simon in random sense.

Simon's conjecture



The multifractal spectrum



Theorem (Jitomirskaya (Ann. 1999), Avila (Arxiv 2008))

The spectral measures of the AMO are absolutely continuous if and only if $|\lambda| < 1$.

The Lebesgue measure of the specturm $|\Sigma_{\alpha,\lambda}| = |4 - 2|\lambda||$.

Theorem (Li-You-Zhou, 2024 Arxiv)

Let $\alpha \in DC$ *and* $0 < \lambda < 1$, *the following results hold:*
(1) If the IDS $\mathcal{N}(x) := \mathcal{N}_{\alpha,\lambda}((-\infty, x]) = k\alpha \mod \mathbb{Z}$, *then*

$$\underline{d}_{\mathcal{N}_{\alpha,\lambda}}(x) = \overline{d}_{\mathcal{N}_{\alpha,\lambda}}(x) = \frac{1}{2}.$$

(2) If $\mathcal{N}(x) \neq k\alpha \mod \mathbb{Z}$, *then*

$$\underline{d}_{\mathcal{N}_{\alpha,\lambda}}(x) \in [1/2, 1]; \quad \overline{d}_{\mathcal{N}_{\alpha,\lambda}}(x) = 1.$$

The multifractal structure of Spectral Measures-AMO

Let $\omega_s(r) := (-\log r)^{-s}$ and define the Diophantine–approximation set

$$D_\alpha(\delta) = \left\{ x \in [0,1] : \limsup_{|k|\to\infty} -\frac{\log \|x - k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} = \delta \right\}.$$

Theorem (C-Li-Wang-Zhou, 2025 Arxiv)

Let $\alpha \in DC$ *and* $0 < \lambda < 1$, *for any* $\kappa \in [1/2, 1), \delta \in (0, \infty]$, *we have*

$$\mathcal{H}^{\omega_s}(D_\alpha(\delta)) = \mathcal{H}^{\omega_s}(\Sigma_{\alpha,\lambda}(\kappa)) = \begin{cases} 0, & \text{if } s > 1, \\ \infty, & \text{if } s \le 1, \end{cases}$$

So $\dim_{H,\log} \Sigma_{\alpha,\lambda}(\kappa) = \dim_{H,\log} D_\alpha(\delta) = 1.$

Thanks for your attention!